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Theory of the Clarinet

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Unlike the flute or the oboe the clarinet yields the twelfth instead of the octave when it is overblown. This is explained by the fact that the mouth-piece owing to its large acoustic impedance behaves as a practically closed end so that only the odd harmonics are present. But Blaikley and Miller [1] have found traces of the second and the fourth harmonics, while the seventh, eighth, ninth and tenth harmonics are unusually strong, the energy-content of the last three being as much as 18%, 15% and 18% respectively of the total intensity. The object of the present paper is to seek an explanation of these anomalies and to describe the general dynamical features of the vibrating system constituting the clarinet.

We regard the body of the instrument as a uniform cylinder of which the axis is taken as the X-axis. The reed supposed to be a light rigid plate of area σ and mass m elastically bound to the end $x=0$. There is a chink between the reed and the mouth-piece, of which the area is a, when the reed is at rest. If the reed is displaced by an amount ζ towards the inside of the mouth-pice, the chink grows smaller and its area becomes $a-\zeta$. Let P be the excess of the pressure of blowing over the atmospheric pressure and p the

[1] D. C. Miller. Science of Musical Sounds, Macmillan & Co. 1916.

pressure due to wave-motion inside the pipe. Then the equation of motion of the reed is

$$m\frac{d^2\zeta}{dt^2}+\mu\zeta=\sigma(P-p) \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where μ is a constant of elasticity.

Let $\phi(ct-x)$ be the velocity-potential of the pulse that initially starts from the mouth-piece down the tube. Then

$$p=\rho\left(\frac{\partial\phi}{\partial t}\right)x=0 \quad \ldots \quad \ldots \quad (2)$$

The rate of total flow through the chink is obviously $(a-\zeta)\left(\frac{\partial\phi}{\partial x}\right)x=0$. We shall assume that this flow is proportional to the difference of pressure set up. Thus,

$$P-p=\omega(a-\zeta)\left(\frac{\partial\phi}{\partial x}\right)x=0 \quad \ldots \quad \ldots \quad (3)$$

where ω is a constant of the nature of acoustic impedance.

From (2) and (3) we get

$$P-\rho c\phi'(ct)=-\omega(a-\zeta)\phi'(ct) \quad \ldots \quad \ldots \quad (4)$$

or $$\phi'(ct)=\frac{P}{\rho c-\omega a} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

if ζ is neglected in comparison with a.

Integrating (5) we get

$$\phi(ct-x)=\kappa P(ct-x) \quad \ldots \quad (6)$$

where $$\kappa=1/(\rho c-\omega a) \quad \ldots \quad \ldots \quad (7)$$

Let l be the length of the pipe. When $\phi(ct-x)$ reaches the end $x=l$, it is reflected as $f(ct+x)$, which is given by the condition that the end in question is an antinode, so that

$$f(ct+x)=-\phi(ct+x-2l)=-\kappa P(ct+x-2l) \quad \dots \quad (8)$$

The pulse $f(ct+x)$ reaches the mouth-piece at the instant $t=2l/c$, and modifies the form of $\phi(ct-x)$ which is to be determined afresh from the condition (3) *viz.*

$$P-\rho c[\phi'(ct)+f'(ct)]=-\omega(\dot{a}-\zeta)\ [\phi'(ct)-f'(ct)]$$

$$\text{or} \quad \phi'(ct)=\kappa P-\chi f'(ct) \quad \dots \quad (9)$$

where $$\chi=\frac{\rho c+\omega\bar{a}}{\rho c-\omega\dot{a}} \quad \dots \quad \dots \quad (10)$$

Making use of (8) in (9) we get

$$\phi'(ct)=\kappa P+\chi\kappa P=(1+\chi)P$$

or $$\phi(ct-x)=(1+\chi)\kappa P(ct-x-2l) \quad \dots \quad (11)$$

the constant of integration being suitably adjusted.

$$f(ct+x)=-(1+\chi)\kappa P(ct+x-4l) \quad \dots \quad \dots \quad (12)$$

The pulse (12) reaches the mouth-piece at the instant $t=4l/c$, when $\phi(ct-x)$ is to be again determined from (9). Thus,

$$\phi'(ct)=\kappa P+\chi(1+\chi)\kappa P=\kappa P(1+\chi+\chi^2)$$

or $$\phi(ct-x)=\kappa P(1+\chi+\chi^2)(ct-x-4l) \quad \dots \quad (13)$$

A study of the forms of ϕ as given by (6), (11) and (13) shows that in the interval between $t=r.2l/c$ and $t=(r+1)2l/c$, we shall have

$$\phi(ct-x)=\kappa P(1+\chi+\chi^2\dots\dots\chi^r)(ct-x-r.2l) \quad \dots \quad (14)$$

Now it is obvious from (10) that if we regard the resistance-factor ω as a very large quantity, then $\chi=-1$ nearly. Thus we have

$$\phi(ct-x)=\kappa P(ct-x), \text{ if } 0<ct<2l,$$

$$\phi(ct-x)=(1+\chi)\kappa P(ct-x-2l), =0, \text{ if } 2l<ct<4l,$$

$$\phi(ct-x)=(1+\chi+\chi^2)\kappa P(ct-x-4l), =\kappa P(ct-x-4l), \text{ if } 4l<ct<6l,$$

$$\phi(ct-x)=(1+\chi+\chi^2+\chi^3)\kappa P(ct-x-6l), =0, \text{ if } 6l<ct<8l,$$

and so on. Similar expressions give $f(ct+x)$. It is obvious that the motion of the system is periodic, the period being given $t=4l/c$, *i.e.*, corresponding to the fundamental period of a pipe closed at one end.

We shall next study the motion at any specified point, say $x=0$, during the interval between $t=2l/c$ and $t=6l/c$. In the interval defined by $2l<ct<4l$, the value of the velocity-potential is the sum of (11) and (8) and equals $\chi\kappa P(ct-2l)$. In the interval $4l<ct<6l$, the velocity-potential is the sum of (13) and (12) and equals $\chi^2\kappa P(ct-4l)$. We shall now expand the velocity-potential as a Fourier series between $ct=2l$ and $ct=6l$.

If we put $ct=4l+\frac{2lz}{\pi}$, then our task is to find a Fourier series which represents

$$\chi\kappa P\left(2l+\frac{2lz}{\pi}\right) \text{ between } z=-\pi \text{ and } z=0,$$

$$\text{and } \chi^2\kappa P\frac{2lz}{\pi} \quad \text{between } z=0 \text{ and } z=\pi$$

Let the series be $\sum a_r \sin rz + \sum b_r \cos rz$.

Then, $$a_r = \frac{1}{\pi} \int_{-\pi}^{o} \chi\kappa P.2l \left(1 + \frac{z}{\pi}\right) \sin rz dz$$

$$+ \frac{1}{\pi} \int_{o}^{\pi} \chi^2 \kappa P.2l \quad \frac{2lz}{\pi} \sin rz dz.$$

Hence, $$\left.\begin{aligned} a_r &= \frac{2l\kappa P}{\pi r}(1+\chi)\chi, \text{ if } r \text{ is even} \\ &= \frac{2l\kappa P}{\pi r}(\chi - 1)\chi, \text{ if } r \text{ is odd.} \end{aligned}\right\} \quad \ldots \quad (15)$$

Remembering that $1 - \chi = -2$, and $1 + \chi = 0$, nearly, we find from (15) that the odd harmonics predominate although the even harmonics are not entirely absent, in agreement with Miller's observations.

So far we have neglected the coupling action between the reed and the pipe. In order to understand the unusual intensity of some of the higher harmonics we shall have to take this coupling into account. If only the first power of ζ is retained in (4) we get instead of (5),

$$\phi'(ct) = \kappa P(1 - \kappa\omega\zeta) \quad \ldots \quad \ldots \quad (16)$$

Putting (16) in (2) and then in (1) we get

$$m\ddot{\zeta} + \mu\zeta = \sigma P - \sigma\rho c.\kappa P(1 - \omega\kappa\zeta) = \sigma P(1 - \rho c\kappa) - \sigma\rho c.\kappa^2 P_{\omega}.\zeta.$$

or $$m\zeta + (\mu - \sigma\rho c\kappa^2 P\omega)\zeta = \sigma P(1 - \rho c\kappa)$$

Its solution is of the form

$$\zeta=\zeta_0+A\cos nt, \qquad \dots \qquad \dots \qquad \dots \quad (17)$$

where
$$n^2=\mu-\sigma\rho c P\omega\kappa^2 \qquad \dots \qquad \dots \quad (18)$$

It is obvious from (18) that the natural frequency of the reed is slightly altered by the coupling action.

If the value of ζ in (17) be put in (16), we find that $\phi(ct-x)$ contains a term of the form B $\sin\ n(t-x/c)$. The contribution of such a term to the Fourier-coefficient a_r considered in (15) will be in the form of the following integral

$$\int_0^\pi \sin rz \sin\left(\frac{n}{c}\cdot\frac{2lz}{\pi}\right)=(-1)^s \sin s\pi.\frac{2s}{r^2-s^2} \qquad \dots \quad (19)$$

where
$$s=\frac{2l}{\pi}.\frac{n}{c} \qquad \dots \qquad \dots \qquad \dots \quad (20)$$

The denominator r^2-s^2 render it obvious that those harmonics of the pipe which nearly agree in frequency with the vibration of the reed will have large amplitudes. Presumably this state of affairs prevails in the neighbourhood of the ninth harmonic. But it is not quite plain why three of them are nearly equal in amplitude. A closer investigation which presently follows will show that the reed vibrates with at least two distinct fundamentals arising out of its peculiar mounting in the mouth-piece.

We have seen that the nature of motion of fluid changes after every half-period of the pipe. For example, the velocity potential at $x=0$, is

$$\kappa P.ct, \text{ if } 0<ct<2l,$$

$$\chi\kappa P(ct-2l), \text{ if } 2l<ct<4l.$$

$$\chi^2\kappa P(ct-4l), \text{ if } 4l<ct<6l,$$

$$\chi^3\kappa P(ct-6l), \text{ if } 6l<ct<8l, \text{ etc.}$$

Hence if p_1, p_2, p_3 etc. denote the values of the pressure p during the intervals defined above, these are given by

$$p_1 = \rho c \kappa P,\; p_2 = \chi p_1,\; p_3 = \chi^2 p_1,\; p_4 = \chi^3 p_1, \text{ etc.}$$

Putting $\chi = -$, we get

$$p_1 = p_3 = p_5 = \quad \dots \quad \dots \quad \dots \quad = \rho c \kappa P,$$

and

$$p_2 = p_4 = p_6 = \quad \dots \quad \dots \quad \dots \quad = -\rho c \kappa P.$$

Thus pressure suffers a discontinuous change at each of the instants $t=0$, $2l/c$, $4l/c$, $6l/c$, etc. Obviously a condensation prevails in the mouth-piece during the first, third, fifth, etc., half-periods, while a rarefaction prevails during the second, fourth, etc.

The equation of motion of the reed during the r^{th} half-period is

$$m\ddot{\zeta} + \mu\zeta = \sigma(P - p_r) = \sigma P\{1 + \rho c \kappa(-1)^r\}$$

or

$$\zeta = \frac{\sigma P}{\mu}\{1 + \rho c \kappa(-1)^r\} + A \cos nt \qquad \dots \qquad \dots \quad (21)$$

The solution (21) can be interpreted as a vibration about a centre of rest which has a sudden shift of amount $\sigma P \rho c \kappa/\mu$ alternately away from and towards the mouth-piece at the instants $t=0$, $2l/c$, $4l/c$, *etc*. Thus during a condensation the reed opens the chink fully so as to admit air into the tube and shuts up during a rarefaction. This tallies well with Raman's[1] description of the mode of excitation and maintenace of the vibration of reed and air-column in the clarinet, which is reproduced below :—

"The reed vibrates practically in the same phase with the change of pressure in the mouth-piece, opening itself so as to admit air into the tube when the pressure is large and shutting up at low pressure."

[1] Raman. "Musik-instrumente und ihre Klänge;" Handbuch der Physik, Vol. 8, p. 410. J. Springer, Berlin.

It is to be noted that when the reed is drawn in, its vibrating length is effectively shortened, since a part of it at the root comes in contact with the mouth-piece, while it vibrates more freely during a condensation. Thus the value of n in (21) is different in the two different half-periods, say they are n_1 and n_2 and let $n_2 > n_1$. Thus there are at least two terms of the form (19) in the expression for the Fourier-co-efficients a_r. Miller's results can be well understood from (20), if we assume that $2ln_1/\pi c$ is slightly greater than 8, and $2ln_2/\pi c$ is slightly less than 10.

My thanks are due to Sir C. V. Raman for having suggested the problem a few years ago and indicated the lines along which the investigation has been carried out.